Influence of Cantilever Vibration on the Reading of a Two-Axis Gyrocompass

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A theoretical and experimental investigation of the influence of cantilever vibration on the reading of a two-axis gyrocompass with spherical air bearings is presented.[1]

A GEOGRAPHICAL coordinate $O\xi\eta\zeta$ is used to derive the equations of motion of a two-axis gyrocompass with spherical air bearings (Fig. 1). Let the ξ axis be directed along the meridional line in the north, the ζ axis vertically to the zenith, and the η axis such that we obtain a right-handed coordinate system.

For simplicity in further investigations, let us assume that the axis of precession, that is, the axis passing through the centers of the spherical air bearings, is placed vertically, and that the center of gravity of the sensitive element coincides with the point of support.

Let us connect the $Oxyz$ coordinate system with the sensitive element and direct the y axis along the axis which passes through the centers of the ball journals (let us call it the axis of rotation of the sensitive element). We shall direct the z axis along the vector of the kinetic moment, assuming that it is perpendicular to the y axis.

The position of the sensitive element relative to the $O\xi\eta\zeta$ coordinate system is determined by angles α, β, and γ, and its motion is determined by the angular velocities $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ (Fig. 2).

The cosines of the small angles between the axes of the $O\xi\eta\zeta$ and $Ox'y'z$ coordinate systems are shown in Table 1.

The projections of the angular velocities on the x', y', and z axes, respectively, will be the following:

$$p' = -\dot{\beta} - U\cos\varphi\cdot\alpha_1$$
$$q' = \dot{\alpha} + U\sin\varphi - U\cos\varphi\cdot\beta$$
$$r = \dot{\gamma} + U\cos\varphi + U\sin\varphi\cdot\beta$$

Let us assume that the moments of inertia of the sensitive element relative to the main axes x, y, and z are B, A, and C. Then the moments of inertia, formed during motion of the sensitive element around the $x'y'z$ axes, can be assumed equal to $B\dot{\beta}$, $-A\dot{\alpha}$, and $-C\dot{\gamma}$, respectively.

The gyroscopic moments relative to the x' and y' axes will be

$$-H(\dot{\alpha} + U\sin\varphi - U\cos\varphi\cdot\beta) \text{ and } -H(\dot{\beta} + U\cos\varphi\cdot\alpha)$$

When the sensitive element turns around the x' and z axes, an elastic precession torque is generated which is equal to the product of the rigidity c of the bearings of the axis of precession and the magnitude of the angle, that is, $c\beta$ and $c\gamma$.

Friction acts on the sensitive element. The moments of the friction forces relative to the x', y', and z axes will be assumed to be proportional to the relative velocity about the inner axis, namely:

$$M_{px'} = k\dot{\beta} \qquad M_{py'} = -k\dot{\alpha} \qquad M_{pz} = -k\dot{\gamma}$$

where k is the coefficient of friction.

In future investigations, we shall take the cantilever vibration and the vibration of the body of the instrument as identical. As a result of the body's vibration, a torque, varying according to a periodic law, the period of which is equal to the period of the cantilever's vibration, will act on the sensitive element. However, only that component of the torque which is perpendicular to its axis of precession will act on the sensitive element. This component of the torque along the axis of precession does not, for all practical purposes, act on the sensitive element, because the friction between the sensitive element and the body of the instrument is negligible.

Let torque M, which is applied to the sensitive element in the plane perpendicular to the axis of precession, vary according to the sinusoidal law $M = m \sin pt$ and its vector form angle θ with the η axis. Then the projection of torque M on the x' and z axes will be

$$M_{x'} = m \sin pt \cos(\theta + \alpha)$$
$$M_z = m \sin pt \sin(\theta + \alpha)$$

or, assuming that angle α is small, we have

$$M_{x'} = m_1 \sin pt - m_2 \sin pt\alpha$$
$$M_z = m_2 \sin pt + m_1 \sin pt\alpha$$

where

$$m_1 = m\cos\theta \qquad m_2 = m\sin\theta$$

If the D'Alambert principle is used and the small value of $Hu\cos\varphi\cdot\beta$ is ignored, then we obtain differential equations of motion for a two-axis gyrocompass on a vibrating cantilever:

$$A\ddot{\alpha} + k\dot{\alpha} + HU\cos\varphi\cdot\alpha + H\dot{\beta} = 0$$
$$B\ddot{\beta} + k\dot{\beta} + c\beta - H\dot{\alpha} - m_2 \sin pt\cdot\alpha = HU\sin\varphi - m_1 \sin pt$$
$$C\ddot{\gamma} + k\dot{\gamma} + c\gamma = m_2 \sin pt + m_1 \sin pt\cdot\alpha$$

From the system of equations obtained it follows that oscillation of the sensitive element relative to the z axis will not influence its behavior in the azimuth. Therefore, in future investigations, we shall not consider the motion of the sensitive element along coordinate γ, and the equations of the two-axis gyrocompass will be as follows:

$$A\ddot{\alpha} + k\dot{\alpha} + HU\cos\varphi\cdot\alpha + H\dot{\beta} = 0$$
$$B\ddot{\beta} + k\dot{\beta} + c\beta - H\dot{\alpha} - m_2 \sin pt\cdot\alpha = HU\sin\varphi = m_1 \sin pt \tag{1}$$

In order to exclude terms describing nutation oscillations in the solution, we omit components $B\ddot{\beta}$ and $k\dot{\beta}$; then Eq. (1) takes the form

$$A\ddot{\alpha} + k\dot{\alpha} + HU\cos\varphi\cdot\alpha + H\dot{\beta} = 0$$
$$c\beta - H\dot{\alpha} - m_2 \sin pt\cdot\alpha = HU\sin\varphi - m_1 \sin pt \tag{2}$$

If the value of β is determined from the second equation in system (2) and substituted into the first equation, we obtain

$$\ddot{\alpha} + \left(2h + \frac{Hm_2}{Ac + H^2}\sin pt\right)\dot{\alpha} + \left(\mu^2 + - \frac{Hpm_2}{Ac + H^2}\cos pt\right)\alpha = \frac{Hpm_1}{Ac + H^2}\cos pt \tag{3}$$

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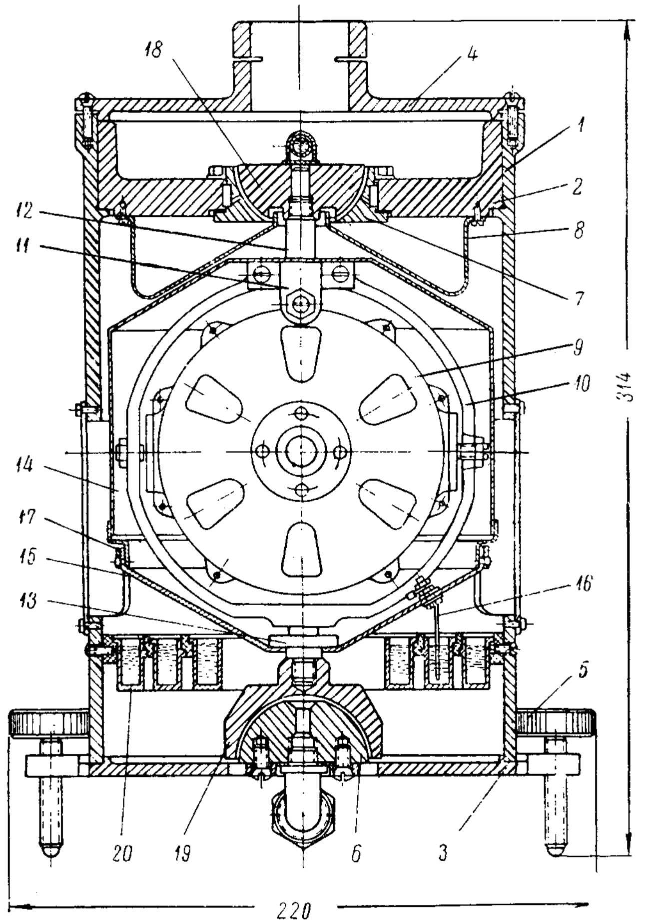

Fig. 1 Drawing of a two axis gyrocompass model: 1) body; 2) upper plate; 3) lower plate; 4) cover; 5) elevating screws; 6) lower ball bearing; 7) upper ball bearing; 8) reflector; 9) gyromotor; 10) vertical ring; 11) cantilever; 12) upper semi-axis; 13) lower semi-axis; 14) housing; 15) lower receptacle; 16) laminated electrode; 17) compass card; 18) upper ball journal; 19) lower ball journal; 20) circular tank

where

$$h = [kc/2(Ac + H^2)]$$

is the coefficient of damping of the sensitive element with no vibration, and

$$\mu^2 = [cHU \cos\varphi/(Ac + H^2)]$$

is the square of the frequency of free undamped oscillations when there is no vibration.

Differential equation (3) cannot be solved in closed form; therefore let us use one of the approximate methods. We introduce

$$\nu = \frac{H}{Ac + H^2} \tag{4}$$

where ν is a small parameter. Taking Eq. (4) into account, Eq. (3) takes the form

$$\ddot{\alpha} + (2h + \nu m_2 \sin pt)\dot{\alpha} + (\mu^2 + \nu m_2 p \cos pt)\alpha = \nu m_1 p \cos pt \tag{5}$$

Let us seek the solution of Eq. (5) as a series expansion in powers of small parameters ν, where the expansion is limited

Table 1

	ξ	η	ζ
x'	$-\alpha$	1	0
y'	$-\beta$	0	1
z	1	α	β

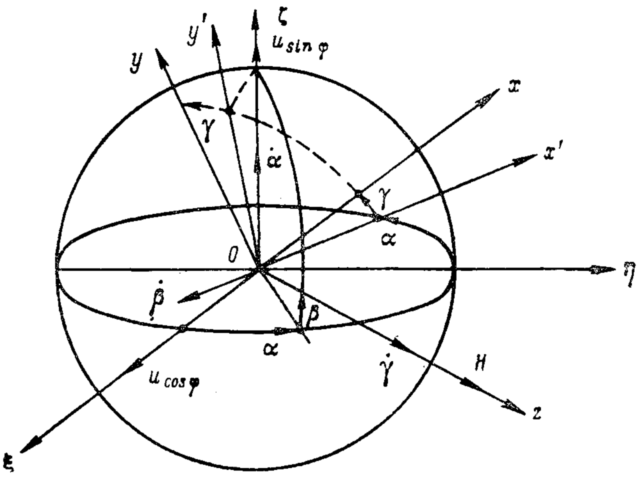

Fig. 2 Angles which determine the position of the sensitive element

only to the terms containing parameter ν to the first power, that is

$$\alpha = \alpha_0 + \nu\alpha_1 + \ldots \tag{6}$$

Substituting series (6) into Eq. (5), we obtain

$$\ddot{\alpha}_0 + 2h\dot{\alpha}_0 + \mu^2\alpha_0 + \nu(\ddot{\alpha}_1 + 2h\dot{\alpha}_1 + \mu^3\alpha_1 + \dot{\alpha}_0 m_2 \sin pt + \alpha_0 m_2 p \cos pt - m_1 p \cos pt) + \ldots = 0 \tag{7}$$

Since Eq. (7) is satisfied for any value of parameter ν, we must have

$$\ddot{\alpha}_0 + 2h\dot{\alpha}_0 + \mu^2\alpha_0 = 0 \tag{8}$$

$$\ddot{\alpha}_1 + 2h\dot{\alpha}_1 + \mu^2\alpha_1 = m_1 p \cos pt - \dot{\alpha}_0 m_2 \sin pt - \alpha_2 m_2 p \cos pt \tag{9}$$

Let us set the initial conditions. At $t = 0$ let

$$\alpha_0(0) + \gamma\alpha_1(0) + \ldots = \alpha_c$$

$$\dot{\alpha}_0(0) + \gamma\dot{\alpha}_1(0) + \ldots = 0$$

For equations of the zero and first approximation, the initial conditions will be as follows:

$$\alpha_0(0) = \alpha \qquad \dot{\alpha}_0 = 0$$
$$\alpha_1(0) = 0 \qquad \dot{\alpha}_1(0) = 0 \tag{10}$$

After minor transformations, we obtain the particular solution of the equation for the zero approximation at $\mu > h$:

$$\alpha_0 = \alpha_c e^{-kt}\left[\cos nt + \frac{h}{n}\sin nt\right] \tag{11}$$

where

$$n = \sqrt{\mu^2 - h^2}$$

Substituting expression (11) into Eq. (9) and making several transformations, we have

$$\dot{\alpha}_1 + 2h\dot{\alpha}_1 + \mu^2\alpha_1 = m_1 p \cos pt + \frac{\alpha_c m_2}{2} e^{-ht} \times \{M \cos[(n - p)t + \psi] - N \cos[(n + p)t - \delta]\}$$

where

$$M = \frac{\mu}{n}\sqrt{\mu^2 - 2pn + p^2}$$

$$N = \frac{\mu}{n}\sqrt{\mu^2 + 2pn + p^2}$$

$$\tan\psi = \frac{ph}{\mu^2 - np} \qquad \tan\delta = \frac{ph}{\mu^2 + np}$$

The particular solution of the equation for the first approximation will be

$$\alpha_1 = e^{-ht}(C_1 \cos nt + C_2 \sin nt) + \frac{m_1 p \cos(pt - c)}{\sqrt{(\mu^2 - p^2)^2 + 4p^2h^2}} + \frac{\alpha_1 m_2}{2p} e^{-ht} \left\{ \frac{M}{2n - p} \cos[(n - p)t + \psi] + \frac{N}{2n + p} \cos[(n + p)t - \delta] \right\}$$

in this

$$C_1 = -\frac{m_1 p \cos\sigma}{\sqrt{\mu^2 - p^2)^2 + 4p^2h^2}} - \frac{\alpha_c m_2}{2p}\left[\frac{M}{2n - p}\cos\psi + \frac{N}{2n + p}\cos\delta\right]$$

$$C_2 = \frac{1}{n}\left\{\frac{(\alpha_c m_2}{2p}\left[\frac{M(n - p)}{2n - p}\sin\psi - \frac{N(n + p)}{2n + p}\sin\delta\right] - \frac{m_1 p(h\cos\delta + p\sin\delta)}{\sqrt{(\mu^2 - p^2)^2 + 1p^2h^2}}\right\}$$

$$\tan\sigma = \frac{2hp}{\mu^2 - p^2}$$

If quantities of the second order of smallness are ignored, and taking into account only terms containing parameter ν to the second power, the solution of Eq. (5) will have the form

$$\alpha = e^{-ht}\left\{(\alpha_c + \nu C_1)\cos nt + \left(\alpha_c \frac{h}{n} + \nu C_2\right)\sin nt + \frac{\alpha_c m_2 \nu}{2p}\left[\frac{M}{2n - p}\cos[(n - p)t + \psi] + \frac{N}{2n - p}\cos[(n + p)t - \delta]\right]\right\} + \frac{m_1 p\nu \cos(pt - c)}{\sqrt{(\mu^2 - p^2)^2 + 4p^3h^2}}$$

From the first approximation, it follows that, if damping of the sensitive element is provided in the instrument, the oscillations, described by terms in the braces, will be damped. After damping of the natural oscillations of the sensitive element, the z axis will execute only forced oscillations, which are determined by the expression

$$\frac{m_1 p\nu \cos(pt - c)}{\sqrt{(\mu^2 - p^2)^2 + 4p^2h^2}}$$

The magnitude of the forced oscillations is a function of the amplitude of torque m, which is generated during centilever vibration, and angle θ, and also the ratio of frequencies μ and p. If the frequency of the natural undamped oscillations of the sensitive element is equal to or near the cantilever's oscillation frequency, then resonance is developed in the system, and the amplitude of the oscillation is somewhat increased.

We shall take a numerical example in order to estimate the magnitude of forced oscillations, which introduce an error into the reading of a two-axis gyrocompass. Now, let us determine the amplitude of the forced oscillations for a two-axis gyrocompass with a spherical air bearing (see Fig. 1). The instrument's parameters are the following: $A = 53$ Gcmsec², $H = 27{,}000$ Gcmsec², $c = 11.6 \cdot 10^6$ Gcm/rad, $T = 70$ sec, $k = 0.55$ Gcmsec, and latitude $\varphi = 60°$.

Let us assume that the cantilever vibrates with a frequency of 30 cps during which the instrument's body turns angularly through $\pm 0'7$. As a result, the torque $m = 23.2$ Gcm at $\theta = 90°$ is applied to the sensitive element through the air bearings. Then the calculated value of the forced oscillation amplitude in azimuth will be $\alpha = 0.'9$.

The numerical results obtained show that cantilever vibration with a frequency of several cps and higher has very little influence on the accuracy of the reading of a two-axis gyrocompass. This assumption is also supported by experimental results. Figure 3 shows two curves of damped oscillations of the sensitive element in azimuth without (Fig. 3a) and with (Fig. 3b) cantilever vibration.

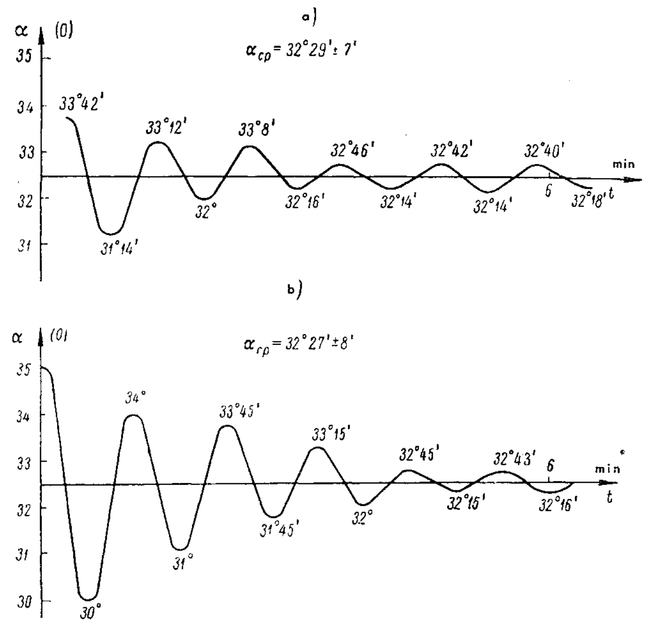

Fig. 3 Curves of sensitive element oscillation in azimuth: a) on a stationary cantilever; b) during cantilever vibration (amplitude, 0₁′, 7; frequency, 30 cps)

These assumptions are correct if the doubled frequency of the damped oscillations of the sensitive element $2n$ is not equal to the cantilever vibration frequency p. If $2n$ is equal to p, then there will be parametric resonance in the system. In this case, motion will be expressed differently—the amplitude of the sensitive element's oscillation, as determined by the ratio of the coefficient of damping to the torque generated during cantilever vibration, increases; that is, the system is unstable.

For the two-axis gyrocompass under investigation with spherical air bearings, the doubled frequency of free damped oscillations $2n$ is approximately equal to $2.86 \cdot 10^{-2}$ cps. During cantilever vibration at a frequency of several cps and higher, there can be no parametric resonance.

The investigations performed are based on the hypothesis that a cantilever vibrates at constant amplitude and frequency. However, the behavior of the sensitive element is essentially unchanged if the cantilever's amplitude and frequency of vibration do not remain constant (Fig. 4). During such vibration, the sensitive element leaves the meridian and starts irregular vibrations. Here natural oscillations of the sensitive element are apparently not damped and are superimposed on forced oscillations. This problem demands special theoretical investigation.

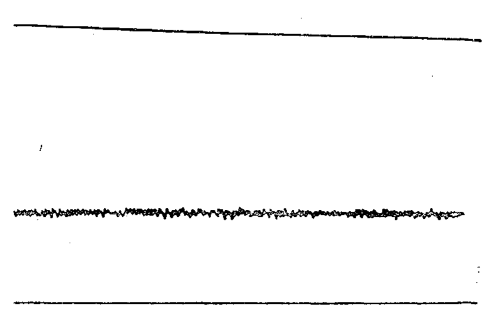

Fig. 4 Curve of irregular cantilever oscillations

Conclusions

1) Theoretical and experimental investigations show that it can be assumed that cantilever vibration at constant frequency and amplitude of several cps and higher does not, for all practical purposes, influence the reading of a two-axis gyrocompass, meaning that, in a real model, its period T will not be less than 30 sec.

2) Experimental observations show that cantilever oscillation at a variable frequency and amplitude distorts the reading of a two-axis gyrocompass and, for all practical purposes, it becomes useless for determining the position of the meridian plane.

—Submitted December 7, 1960

Reference

[1] Il'in, P. A. and Sergeyev, M. A., "A terrestrial two-axis gyrocompass with spherical air bearings. Problems of the theory and calculation of gyroscopic instruments and precision instruments", LITMO (Leningrad Institute of Precision Mechanics and Optics), no. 36 (1958).

Reviewer's Comment

The reviewer considers Fig. 1 the primary asset of this paper, since it tells us quite a bit about the state of the art of Russian gyro technology.

In addition, it is of interest to follow the mathematical derivations, which appear to be correct. The reviewer arrives at the conclusion that the experimental observations, which are in themselves revealing, are not adequately explained by the theoretical analysis.

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Method of Superposition under Conditions of Elasticity and Destructive Stress

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THIS note is intended to show that, given the ordinary assumptions about the connection between stresses and deformations in a system consisting of a plane and a direction, the method of superposition, analogous to the Batdorf-Budiansky method in the theory of slip,[1] can be used to obtain the ordinary relations of elasticity. This result, which confirms the validity of the theory of slip, is also of interest in that it evidently brings out a new aspect of the problem of the correlation between the theoretical and observed stresses of brittle fracture.

Let us take a given deformation ϵ_{ij} in the system of axes x, y, z. Then an element with normal n will experience the normal deformation

$$\epsilon_{nn} = l_{in}l_{jn}\epsilon_{ij} \tag{1}$$

and in the direction m a shear deformation

$$\epsilon_{nm} = l_{in}l_{jm}\epsilon_{ij} \tag{2}$$

Here l_{in}, l_{im} are the direction cosines in the system of axes x, y, z of the normal n and the direction m, respectively. As a result of this, the element with normal n is subjected to a normal stress σ_{nn} and a tangential stress σ_{nm} in the direction m. In the axes x, y, z the stresses σ_{nn} and σ_{nm} produce stresses

$$\sigma_{ij}^{\circ} = \sigma_{nn}l_{in}l_{jn} + \sigma_{nm}(l_{in}l_{jm} + l_{im}l_{jn}) \tag{3}$$

Assuming that the total stress σ_{ij} in the axes x, y, z is the average of these unit stresses with respect to all possible surface elements and directions within them (this assumption coincides with the procedure in the theory of slip except insofar as stresses are substituted for deformations), we get

$$\sigma_{ij} = \frac{1}{Q_1}\int_{Q_1} \sigma_{nn}l_{in}l_{jn}dQ_1 + \frac{1}{Q_2}\int_{Q_2} \sigma_{nm}(l_{in}l_{im} + l_{jm}l_{jn})dQ_2 \tag{4}$$

$$dQ_1 = d\Omega \qquad dQ_2 = d\Omega d\beta \tag{5}$$

Here Ω is the solid angle, and β is the angle formed by the direction m at the element with normal n and a certain fixed direction.

The direction cosines l_{in} and l_{im} can easily be expressed in terms of longitude α and latitude φ on a unit sphere and the angle β, as in the theory of slip:[1]

$$\begin{aligned}
l_{xn} &= \sin\alpha\,\cos\varphi \\
l_{yn} &= \cos\alpha\,\cos\varphi \\
l_{zn} &= \sin\varphi \\
l_{xm} &= \cos\alpha\,\sin\beta - \sin\alpha\,\cos\beta\,\sin\varphi \\
l_{ym} &= -\sin\alpha\,\sin\beta - \cos\alpha\,\cos\beta\,\sin\varphi \\
l_{zm} &= \cos\beta\,\cos\varphi
\end{aligned} \tag{6}$$

where

$$d\Omega = \cos\varphi d\varphi d\alpha \tag{7}$$

In investigating the validity of this procedure for obtaining the connection between stresses and deformations in the elastic case, it is natural to assume that in a system consisting of a plane and a direction the stresses and deformations are connected by the relations

$$\sigma_{nn} = a\epsilon_{nn} \qquad \sigma_{nm} = b\epsilon_{nm} \qquad (a,b = \text{const}) \tag{8}$$

As distinct from the case of plastic deformations, in the case in question integration must be carried out with respect

Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki (Journal of Applied Mechanics and Technical Physics), no. 3, 103–104 (1961). Translated by Faraday Translations, New York.

[1] Batdorf, S. B. and Budiansky, B. A., "Mathematical theory of plasticity based on the concept of slip," NACA TN 1871 (April 1949).